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Factorization theorems for high energy nn, γp and $\gamma\gamma$ scattering*

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Abstract. The robustness of the factorization theorem for total cross sections, $\sigma_{nn}/\sigma_{\gamma p} = \sigma_{\gamma p}/\sigma_{\gamma\gamma}$ for nn (the *even* portion of pp and $\bar{p}p$ scattering), γp and $\gamma\gamma$ scattering, originally proved by Block and Kaidalov using an eikonal formalism, is demonstrated. Factorization theorems for the nuclear slope parameter B and ρ, the ratio of the real to the imaginary portion of the forward scattering amplitude, are derived under very general conditions, using analyticity and the optical theorem.

Recently, Block and Kaidalov [1] have proved three high energy factorization theorems:

$$\frac{\sigma_{nn}(s)}{\sigma_{\gamma p}(s)} = \frac{\sigma_{\gamma p}(s)}{\sigma_{\gamma\gamma}(s)}, \tag{1}$$

where the σs are the total cross sections for nucleon-nucleon, γp and $\gamma\gamma$ scattering;

$$\frac{B_{nn}(s)}{B_{\gamma p}(s)} = \frac{B_{\gamma p}(s)}{B_{\gamma\gamma}(s)}, \tag{2}$$

where the Bs are the nuclear slope parameters for elastic scattering, and

$$\rho_{nn}(s) = \rho_{\gamma p}(s) = \rho_{\gamma\gamma}(s), \tag{3}$$

where the ρs are the ratio of the real to imaginary portions of the forward scattering amplitudes, with the first two factorization theorems each having their own proportionality constant. In the above, nn (for nucleon-nucleon) denotes the *even* portion of the pp and $\bar{p}p$ scattering amplitude. The even nucleon-nucleon portion is required since the photon is even under crossing. Their derivation assumed a eikonal model, with the (complex) eikonal $\chi(b,s)$ such that $a(b,s)$, the (complex) scattering amplitude in impact parameter space b, is given by

$$\begin{aligned} a(b,s) &= \frac{\mathrm{i}}{2}\left(1-\mathrm{e}^{\mathrm{i}\chi(b,s)}\right) \\ &= \frac{\mathrm{i}}{2}\left(1-\mathrm{e}^{-\chi_{\mathrm{I}}(b,s)+\mathrm{i}\chi_{\mathrm{R}}(b,s)}\right). \end{aligned} \tag{4}$$

Using the optical theorem, the total cross section $\sigma_{\mathrm{tot}}(s)$ is given by

$$\begin{aligned} \sigma_{\mathrm{tot}}(s) &= 4\int \mathrm{d}^2\mathbf{b}\,\mathrm{Im}\,a(b,s) \\ &= 2\int \left\{1-\mathrm{e}^{-\chi_{\mathrm{I}}(b,s)}\cos[\chi_{\mathrm{R}}(b,s)]\right\}\mathrm{d}^2\mathbf{b}, \end{aligned} \tag{5}$$

and the elastic scattering cross section $\sigma_{\mathrm{el}}(s)$ is given by

$$\begin{aligned} \sigma_{\mathrm{el}}(s) &= 4\int \mathrm{d}^2\mathbf{b}\,|a(b,s)|^2 \\ &= \int \left|1-\mathrm{e}^{-\chi_{\mathrm{I}}(b,s)+\mathrm{i}\chi_{\mathrm{R}}(b,s)}\right|^2 \mathrm{d}^2\mathbf{b}. \end{aligned} \tag{6}$$

The ratio of the real to the imaginary portion of the forward nuclear scattering amplitude, $\rho(s)$, is given by

$$\rho(s) = \frac{\mathrm{Re}\int \mathrm{d}^2\mathbf{b}\,a(b,s)}{\mathrm{Im}\int \mathrm{d}^2\mathbf{b}\,a(b,s)}, \tag{7}$$

and the nuclear slope parameter $B(s)$ is given by

$$B(s) = \frac{\int b^2 a(b,s)\,\mathrm{d}^2\mathbf{b}}{2\int a(b,s)\,\mathrm{d}^2\mathbf{b}}. \tag{8}$$

They used an even (under crossing) QCD-inspired eikonal χ^{even} for nn scattering, given by the sum of three contributions, glue-glue, quark-glue and quark-quark, which are individually factorizable into a product of a cross section $\sigma(s)$ times an impact parameter space distribution function $W(b\,;\mu)$, i.e.,

$$\begin{aligned} \chi^{\mathrm{even}}(s,b) &= \chi_{gg}(s,b)+\chi_{qg}(s,b)+\chi_{qq}(s,b) \\ &= \mathrm{i}\big[\sigma_{gg}(s)W(b\,;\mu_{gg})+\sigma_{qg}(s)W(b\,;\mu_{qg}) \\ &\quad +\sigma_{qq}(s)W(b\,;\mu_{qq})\big]. \end{aligned} \tag{9}$$

The impact parameter space distribution functions used in (9) were taken to be the convolution of two dipole form factors, i.e.,

$$W(b\,;\mu) = \frac{\mu^2}{96\pi}(\mu b)^3 K_3(\mu b), \tag{10}$$

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where $K_3(x)$ is a modified Bessel function. For large s, the even amplitude in (9) is made analytic by the substitution $s \to s\mathrm{e}^{-\mathrm{i}\pi/2}$ (see the table on p. 580 of [9]).

They required that the ratio of elastic to total scattering be process independent, i.e.,

$$\left(\frac{\sigma_{\rm el}}{\sigma_{\rm tot}}\right)^{nn} = \left(\frac{\sigma_{\rm el}}{\sigma_{\rm tot}}\right)^{\gamma p} = \left(\frac{\sigma_{\rm el}}{\sigma_{\rm tot}}\right)^{\gamma\gamma} \tag{11}$$

at *all* energies, a condition that insures that each process becomes equally black disk-like as we go to high energy. As shown in [1], the eikonal $\chi^{\gamma p}(s,b)$ is obtained from $\chi^{\rm even}(s,b)$ by multiplying each σ in $\chi^{\rm even}$ by κ and each μ^2 by $\frac{1}{\kappa}$. Thus, in impact parameter space, $\langle b^2_{\gamma p}\rangle = \kappa\langle b^2_{nn}\rangle$, where the $\langle b^2\rangle$s are the mean squared values of the impact parameter b, i.e.,

$$\langle b^2\rangle = \int b^2 W(b;\mu)\mathrm{d}^2\mathbf{b}, \tag{12}$$

since the distribution $W(b;\mu)$ is normalized so that $1 = \int W(b;\mu)\mathrm{d}^2\mathbf{b}$. In turn, the eikonal $\chi^{\gamma\gamma}(s,b)$ is obtained from $\chi^{\gamma p}(s,b)$ by multiplying each σ in $\chi^{\gamma p}$ by κ and each μ^2 by $\frac{1}{\kappa}$. Thus, factorization requires that

$$\langle b^2_{\gamma\gamma}\rangle = \kappa\langle b^2_{\gamma p}\rangle = \kappa^2\langle b^2_{nn}\rangle, \tag{13}$$

i.e., the mean square radii in impact parameter space, along with the cross sections σ, also scale as κ. We note that the *same* κ must be used in the gluon sector as in the quark sector for the ratio of $(\frac{\sigma_{\rm el}}{\sigma_{\rm tot}})$ to be process independent (see (11)). It should be emphasized that the derivation in [1] assumed that the impact parameter distribution functions $W(b\,;\mu)$ used in the nn, γp and $\gamma\gamma$ eikonals had the *same* functional forms (see (10)) for all three processes nn, γp and $\gamma\gamma$, differing only in the size of the μs.

The purpose of this note is as follows.
(1) Demonstrate the robustness of the original total cross section theorem against the choice of the same functional forms $W(b)$ for the proton and photon, by using *different* functional forms for the $W(b)$ in the eikonals for the three processes nn, γp and $\gamma\gamma$. We will now assume that the photon has a monopole form factor, whereas the nucleon is assumed to have a dipole form factor.
(2) Derive the ρ theorem, (3), using only analyticity.
(3) Derive the nuclear slope theorem, (2), essentially using only the optical theorem.

We will now show the robustness of the theorem of (1),

$$\frac{\sigma_{nn}(s)}{\sigma_{\gamma p}(s)} = \frac{\sigma_{\gamma p}(s)}{\sigma_{\gamma\gamma}(s)}. \tag{14}$$

We no longer assume that all three processes have the same functional form $W(b;\mu)$ in impact parameter space, but now we assume that the photon (like the pion) is best described by a *monopole* form factor, whereas the nucleon is assumed to have a dipole form factor. Here, we are assuming that the transverse matter distributions of the pion and the proton (antiproton) are given by a monopole and dipole form factor, respectively, having the same shape as the charge distribution [2,3]. Thus, the impact parameter space distributions $W(b)$ that are appropriate for nn, γp and $\gamma\gamma$ scattering are

$$W_{22}(b;\mu) = \frac{1}{(2\pi)^2}\int\frac{\mu^8}{(q^2+\mu^2)^4}\mathrm{e}^{\mathrm{i}\mathbf{q}\cdot\mathbf{b}}\mathrm{d}^2\mathbf{q} \tag{15}$$

(dipole-dipole)

$$W_{12}(b;\mu,\nu) = \frac{1}{(2\pi)^2}\int\frac{\mu^4\nu^2}{(q^2+\mu^2)^2(q^2+\nu^2)}\mathrm{e}^{\mathrm{i}\mathbf{q}\cdot\mathbf{b}}\mathrm{d}^2\mathbf{q} \tag{16}$$

(monopole-dipole)

$$W_{11}(b;\nu) = \frac{1}{(2\pi)^2}\int\frac{\nu^4}{(q^2+\nu^2)^2}\mathrm{e}^{\mathrm{i}\mathbf{q}\cdot\mathbf{b}}\mathrm{d}^2\mathbf{q} \tag{17}$$

(monopole-monopole) and are given by

$$W_{22}(b;\mu) = \frac{\mu^2}{96\pi}(\mu b)^3K_3(\mu b), \tag{18}$$

$$W_{12}(b;\mu,\nu) = \frac{\mu^2\nu^2}{2\pi(\mu^2-\nu^2)} \times\left(\frac{\mu^2}{\mu^2-\nu^2}[K_0(\nu b)-K_0(\mu b)] - \frac{\mu b}{2}K_1(\mu b)\right), \tag{19}$$

$$W_{11}(b;\nu) = \frac{\nu^2}{4\pi}(\nu b)K_1(\nu b), \tag{20}$$

where the Ks are modified Bessel functions. These distributions are all normalized such that $\int W(b)\mathrm{d}^2\mathbf{b} = 1$. It is readily shown from (18), (19) and (20) that the mean squared sizes, in impact parameter space, of the dipole-dipole, monopole-dipole and monopole-monopole distributions are given by

$$\langle b^2_{22}\rangle = \frac{16}{\mu^2}, \tag{21}$$

$$\langle b^2_{12}\rangle = \frac{8}{\mu^2} + \frac{4}{\nu^2}, \tag{22}$$

$$\langle b^2_{11}\rangle = \frac{8}{\nu^2}. \tag{23}$$

To fix the parameter ν in (22) and (23), we will require that the distributions $W_{11}(b;\mu)$, $W_{12}(b;\mu,\nu)$ and $W_{11}(b;\nu)$ all have the same mean square size, i.e.,

$$\langle b^2_{22}\rangle = \langle b^2_{12}\rangle = \langle b^2_{11}\rangle = \frac{16}{\mu^2}. \tag{24}$$

This gives us

$$\nu = \frac{1}{\sqrt{2}}\mu. \tag{25}$$

We now introduce the simpler notation

$$W_{22}(b) = W_{22}(b;\mu) = \frac{\mu^2}{96\pi}(\mu b)^3K_3(\mu b), \tag{26}$$

$$W_{12}(b) = W_{12}(b;\mu,\nu=\mu/\sqrt{2}) \tag{27}$$

$$= \frac{\mu^2}{8\pi}\left(4\left\{K_0\left(\frac{\mu b}{\sqrt{2}}\right) - K_0(\mu b)\right\} - (\mu b)K_1(\mu b)\right),$$

$$W_{11}(b) = W_{11}(b;\mu,\nu=\mu/\sqrt{2}) = \frac{\mu^2}{8\pi}\left(\frac{\mu b}{\sqrt{2}}\right)K_1\left(\frac{\mu b}{\sqrt{2}}\right), \quad (28)$$

with the properties that all $W(b)_{ij}$ have the same mean square size and are all normalized distributions. We note that from (13), the requirement of factorization is that

$$\langle b^2_{\gamma\gamma}\rangle = \kappa\langle b^2_{\gamma p}\rangle = \kappa^2\langle b^2_{nn}\rangle. \quad (29)$$

It is straightforward to show that if we let $W_{nn}(b;\mu) = W_{22}(b;\mu)$, form $W_{\gamma p}$ by letting $\mu \to \mu/\sqrt{\kappa}$ in (27) and form $W_{\gamma\gamma}$ by letting $\mu \to \mu/\kappa$ in (28), i.e.,

$$W_{nn}(b;\mu) = \frac{\mu^2}{96\pi}(\mu b)^3 K_3(\mu b), \quad (30)$$

$$W_{\gamma p}(b;\mu) = \frac{1}{\kappa}\frac{\mu^2}{8\pi} \quad (31)$$
$$\times\left(4\left\{K_0\left(\frac{1}{\sqrt{2}}\frac{\mu b}{\sqrt{\kappa}}\right) - K_0\left(\frac{\mu b}{\sqrt{\kappa}}\right)\right\} - \left(\frac{\mu b}{\sqrt{\kappa}}\right)K_1\left(\frac{\mu b}{\sqrt{\kappa}}\right)\right),$$

$$W_{\gamma\gamma}(b;\mu) = \frac{1}{\kappa^2}\frac{\mu^2}{8\pi}\left(\frac{1}{\sqrt{2}}\frac{\mu b}{\kappa}\right)K_1\left(\frac{1}{\sqrt{2}}\frac{\mu b}{\kappa}\right), \quad (32)$$

that these impact parameter distributions for the nn, γp and $\gamma\gamma$ processes satisfy the factorization requirement of (29). To demonstrate this, we make variable substitutions in the definition of $\langle b^2\rangle$,

$$\langle b^2\rangle_i = \int b^2 W_i(b;\mu)\,\mathrm{d}^2\mathbf{b} \quad (33)$$
$$= 2\pi\int_0^\infty b^3 W_i(b;\mu)\,\mathrm{d}b, \quad i = nn,\ \gamma p, \gamma\gamma.$$

For nn, we get $\langle b^2\rangle_{nn} = \langle b^2\rangle_{22}$. For γp, we make the variable change $x = \frac{\mu b}{\sqrt{\kappa}}$ in (33) and find that $\langle b^2\rangle_{\gamma p} = \kappa\langle b^2\rangle_{nn}$. Finally, for $\gamma\gamma$, we make the variable change $x = \frac{\mu b}{\kappa}$ in (33) and find that $\langle b^2\rangle_{\gamma\gamma} = \kappa^2\langle b^2\rangle_{nn}$, thus demonstrating that we satisfy the factorization requirements of (29).

For numerical calculations, we will use $\mu = 0.89\,\mathrm{GeV}$ [6].

We now compare the total cross sections and the ratio of elastic to total cross sections for the three processes, at several representative energies, to investigate factorization numerically.

For simplicity of calculation, we will further assume that the eikonal is purely imaginary ($\rho = 0$) and is composed of only one term,

$$\chi^i(s,b) = \mathrm{i}\Sigma^i(s) \times W_i(b;\mu), \quad (34)$$

with $i = nn$, γp and $\gamma\gamma$, respectively. Thus, we use the Chou-Yang model [2, 3] for our numerical calculations of elastic scattering and total cross section, i.e., an amplitude $a(b,s)$ in impact parameter space

$$a(b,s) = \frac{\mathrm{i}}{2}\left(1 - \mathrm{e}^{-\Sigma^i(s)W_i(b;\mu)}\right). \quad (35)$$

This is a reasonable approximation since the ρ value for nn scattering is known to be small at high energies.

As in [1], the Σ^i are chosen so that they scale in κ, i.e.,

$$\Sigma^{\gamma\gamma}(s) = \kappa\Sigma^{\gamma p}(s) = \kappa^2\Sigma^{nn}(s) \quad (36)$$

($\kappa = 2/3$, the value for the additive quark model, was used in [1]). Using the amplitude of (35) to test the factorization hypothesis of Block and Kaidalov [1], we must evaluate the relations

$$\sigma^i_{\mathrm{tot}} = 2 \times F^i\int\left[1 - \mathrm{e}^{-\Sigma^i(s)W_i(b;\mu)}\right]\mathrm{d}^2\mathbf{b}, \quad (37)$$

$$\left(\frac{\sigma_{\mathrm{el}}}{\sigma_{\mathrm{tot}}}\right)^i = \frac{F^i\int\left[1 - \mathrm{e}^{-\Sigma^i(s)W_i(b;\mu)}\right]^2\mathrm{d}^2\mathbf{b}}{\sigma^i_{\mathrm{tot}}}, \quad (38)$$

where the normalization factors F^i are 1, P^γ_{had} and $(P^\gamma_{\mathrm{had}})^2$ for $i = nn$, γp and $\gamma\gamma$, respectively. P^γ_{had} is the probability that a photon turns into a vector meson. We will use $P^\gamma_{\mathrm{had}} = 1/240$, a value consistent with the vector dominance model [1].

We will evaluate (37) and (38) for values of $\Sigma^{nn}(s) = 160$, 110 and $67\,\mathrm{GeV}^{-2}$, chosen to reproduce the nucleon-nucleon total cross section, $\sigma^{nn}_{\mathrm{tot}}$, at cms energies of ≈ 1500, 400 and 25 GeV, respectively. The results for $\sigma_{\mathrm{el}}/\sigma_{\mathrm{tot}}$, σ_{tot} as a function of energy for the three reactions nn, γp and $\gamma\gamma$, as well as the energy dependence of the total cross sections $\sigma_{nn}/\sigma_{\gamma p}$ and $\sigma_{\gamma p}/\sigma_{\gamma\gamma}$, are given in Table 1.

For *small* eikonals, we can easily show that the total cross sections scale *exactly* as $\kappa P^\gamma_{\mathrm{had}}$, completely independent of the form factor shape. In the limit of a small eikonal, the integral $\int\left[1 - \mathrm{e}^{-\Sigma^i(s)W_i(b)}\right]\mathrm{d}^2\mathbf{b}$ in (37) goes to $\int\Sigma^i(s)W_i(b)\,\mathrm{d}^2\mathbf{b} = \Sigma^i(s)$, since $\int W_i(b)\,\mathrm{d}^2\mathbf{b} = 1$ for all three processes. Thus, $\sigma^{\gamma\gamma}_{\mathrm{tot}} = \kappa P^\gamma_{\mathrm{had}}\sigma^{\gamma p}_{\mathrm{tot}} = (\kappa P^\gamma_{\mathrm{had}})^2\sigma^{\gamma\gamma}_{\mathrm{tot}}$ for small eikonals.

What is most striking is that for the *large* eikonals used in Table 1, the ratios of total cross sections also closely scale as $\kappa P^\gamma_{\mathrm{had}}$, essentially independent of the choice of the form factor shape. It is straightforward, again by using the variable substitution of $x \to \frac{\mu b}{\sqrt{\kappa}}$ for γp processes and the substitution of $x \to \frac{\mu b}{\sqrt{\kappa^2}}$ into (37), that we find

$$\sigma^{nn}_{\mathrm{tot}} = 2 \times \int\left[1 - \mathrm{e}^{-\Sigma^{nn}W_{22}(b;\mu)}\right]\mathrm{d}^2\mathbf{b} = \sigma_{22}(\Sigma^{nn},\mu) \quad (39)$$

$$\sigma^{\gamma p}_{\mathrm{tot}} = \kappa P^\gamma_{\mathrm{had}} \times 2\int\left[1 - \mathrm{e}^{-\Sigma^{nn}W_{12}(b;\mu)}\right]\mathrm{d}^2\mathbf{b} = \kappa P^\gamma_{\mathrm{had}} \times \sigma_{12}(\Sigma^{nn},\mu), \quad (40)$$

$$\sigma^{\gamma\gamma}_{\mathrm{tot}} = \kappa^2 P^\gamma_{\mathrm{had}} \times 2\int\left[1 - \mathrm{e}^{-\Sigma^{nn}W_{22}(b;\mu)}\right]\mathrm{d}^2\mathbf{b} = \kappa^2 P^\gamma_{\mathrm{had}} \times \sigma_{11}(\Sigma^{nn},\mu), \quad (41)$$

where the cross sections $\sigma_{22}(\Sigma^{nn},\mu)$, $\sigma_{12}(\Sigma^{nn},\mu)$, $\sigma_{11}(\Sigma^{nn},\mu)$, defined in (39), (40) and (41), respectively,

Table 1. The energy dependence of $\sigma_{\rm el}/\sigma_{\rm tot}$ and $\sigma_{\rm tot}$ for the three reactions nn, γp and $\gamma\gamma$, and the energy dependence of the ratios $\sigma_{\rm tot}^{nn}/\sigma_{\rm tot}^{\gamma p}$ and $\sigma_{\rm tot}^{\gamma p}/\sigma_{\rm tot}^{\gamma\gamma}$, where $\kappa = 2/3$ and $P_{\rm had}^{\gamma} = 1/240$

Reaction	$\sqrt{s} = 1500$ GeV		$\sqrt{s} = 400$ GeV		$\sqrt{s} = 25$ GeV	
	$\sigma_{\rm el}/\sigma_{\rm tot}$	$\sigma_{\rm tot}$	$\sigma_{\rm el}/\sigma_{\rm tot}$	$\sigma_{\rm tot}$	$\sigma_{\rm el}/\sigma_{\rm tot}$	$\sigma_{\rm tot}$
nn	0.272	73.00 mb	0.236	57.4 mb	0.184	40.06 mb
γp	0.268	199.7 μb	0.235	157.8 μb	0.186	109.9 μb
$\gamma\gamma$	0.263	545.3 nb	0.233	428.5 nb	0.188	300.4 nb
$\sigma_{\rm tot}^{nn}/\sigma_{\rm tot}^{\gamma p}$	$1.015/(\kappa P_{\rm had}^{\gamma})$		$1.015/(\kappa P_{\rm had}^{\gamma})$		$1.013/(\kappa P_{\rm had}^{\gamma})$	
$\sigma_{\rm tot}^{\gamma p}/\sigma_{\rm tot}^{\gamma\gamma}$	$1.017/(\kappa P_{\rm had}^{\gamma})$		$1.018/(\kappa P_{\rm had}^{\gamma})$		$1.016/(\kappa P_{\rm had}^{\gamma})$	

are *independent* of κ. Clearly, any departure from factorization lies in the ratios of $\sigma_{22}(\Sigma^{nn},\mu)/\sigma_{12}(\Sigma^{nn},\mu)$ and $\sigma_{12}(\Sigma^{nn},\mu)/\sigma_{11}(\Sigma^{nn},\mu)$ being not equal to unity, a statement which is *independent* of the choices of $P_{\rm had}^{\gamma}$ and κ. Thus, for example, our results from Table 1 that at 1500 GeV, $\sigma_{\rm tot}^{nn}/\sigma_{\rm tot}^{\gamma p} \approx 1.015 \times \kappa P_{\rm had}^{\gamma}$ and $\sigma_{\rm tot}^{\gamma p}/\sigma_{\rm tot}^{\gamma\gamma} \approx 1.017 \times \kappa P_{\rm had}^{\gamma}$, show that, *independent* of the values of κ and $P_{\rm had}^{\gamma}$, $\sigma_{22}(\Sigma^{nn},\mu)/\sigma_{12}(\Sigma^{nn},\mu) = 1.015$ and that $\sigma_{12}(\Sigma^{nn},\mu)/\sigma_{11}(\Sigma^{nn},\mu) = 1.017$, a very tiny departure from perfect factorization.

Further, we note that the ratios of elastic to total cross section are closely the same for all three processes (although they have an energy dependence), to better than ±1.9%, ±0.9% and ±1.1% at 1500, 400 and 25 GeV, respectively. We see that the ratios $\sigma_{\rm tot}^{nn}/\sigma_{\rm tot}^{\gamma p}$ and $\sigma_{\rm tot}^{\gamma p}/\sigma_{\rm tot}^{\gamma\gamma}$ are *both systematically* ≈ 1.5% too large. As a consequence, the original cross section factorization theorem of Block and Kaidalov [1], $\frac{\sigma_{nn}(s)}{\sigma_{\gamma p}(s)} = \frac{\sigma_{\gamma p}(s)}{\sigma_{\gamma\gamma}(s)}$, is shown numerically to be valid to ≈ 0.3% over a very large range of s. This result also is independent of κ and $P_{\rm had}^{\gamma}$.

Thus, even allowing for the photon to have a monopole form factor, thus loosening the constraint of identical $W(b)$ functional forms for the three processes, the cross section factorization theorem is very robust.

We now turn our attention to the factorization theorems involving ρ and B. We will find $\rho_{\gamma\gamma}(s)$ utilizing an analysis involving real analytic amplitudes, a technique first proposed by Bourrely and Fischer [4] and later utilized extensively by Nicolescu and Kang [5]. We follow the procedures and conventions used by Block and Cahn [9]. The variable s is the square of the CM system energy, whereas ν is the laboratory system momentum. Using the optical theorem, in terms of the *even* laboratory scattering amplitude f_+, where $f_+(\nu) = f_+(-\nu)$, the total even cross section $\sigma_{\rm tot}$ is given by

$$\sigma_{\rm tot} = \frac{4\pi}{\nu}{\rm Im} f_+(\theta_{\rm lab} = 0), \tag{42}$$

where $\theta_{\rm lab}$ is the laboratory scattering angle. We further assume that our amplitudes are real analytic functions with a simple cut structure [9]. We use an even amplitude for reactions in the high energy region, far above any cuts (see [9], p. 587, (5.5a), with $a = 0$), where the even amplitude simplifies considerably and is given, for example, by

$$f_+(s) = {\rm i}\frac{\nu}{4\pi}\Big\{A + \beta\big[\ln(s/s_0) - {\rm i}\pi/2\big]^2 + cs^{\mu-1}{\rm e}^{{\rm i}\pi(1-\mu)/2}\Big\}, \tag{43}$$

where A, β, c, s_0 and μ are real constants. We are ignoring any real subtraction constants. In (43), we have assumed that the total cross section rises asymptotically as $\ln^2 s$. The real and imaginary parts of (43) are given by

$${\rm Re}\frac{4\pi}{\nu} f_+(s) = \beta\,\pi \ln s/s_0 - c\,\cos(\pi\mu/2)s^{\mu-1}, \tag{44}$$

$$\begin{aligned} &{\rm Im}\frac{4\pi}{\nu} f_+(s) \\ &= A + \beta\left[\ln^2 s/s_0 - \frac{\pi^2}{4}\right] + c\,\sin(\pi\mu/2)s^{\mu-1}. \end{aligned} \tag{45}$$

Using (42), (44) and (45), the total cross section for high energy scattering is given by

$$\sigma_{\rm tot}(s) = A + \beta\left[\ln^2 s/s_0 - \frac{\pi^2}{4}\right] + c\,\sin(\pi\mu/2)s^{\mu-1}, \tag{46}$$

and ρ, the ratio of the real to the imaginary portion of the forward scattering amplitude, is given by

$$\begin{aligned} \rho(s) &= \frac{\beta\,\pi\ln s/s_0 - c\,\cos(\pi\mu/2)s^{\mu-1}}{A + \beta\left[\ln^2 s/s_0 - \frac{\pi^2}{4}\right] + c\,\sin(\pi\mu/2)s^{\mu-1}} \\ &= \frac{\beta\,\pi\ln s/s_0 - c\,\cos(\pi\mu/2)s^{\mu-1}}{\sigma_{\rm tot}}. \end{aligned} \tag{47}$$

Clearly, if the cross sections for the three processes, nn, γp, and $\gamma\gamma$ scale, i.e., $\frac{\sigma_{nn}(s)}{\sigma_{\gamma p}(s)} = \frac{\sigma_{\gamma p}(s)}{\sigma_{\gamma\gamma}(s)}$, then by inspection of (46), the coefficients A, β and c for each of the three processes scale. Since ρ, as seen in (47), is a ratio of such terms, all three ρ values are the *same*, i.e.,

$$\rho_{nn}(s) = \rho_{\gamma p}(s) = \rho_{\gamma\gamma}(s), \tag{48}$$

which is the ρ factorization theorem of (3). Clearly, the argument does not depend on the specific form of the amplitude assumed in (43), being equally valid if the cross section were to rise as $\ln s$, or even rise as a power of s.

Let us now turn our attention to the factorization theorem of the nuclear slopes. We will assume that the differential elastic scattering is adequately parameterized by

$$\frac{d\sigma_{el}}{dt} = \left[\frac{d\sigma_{el}}{dt}\right]_{t=0} e^{Bt}. \quad (49)$$

We can write, working in the center of mass system,

$$\left[\frac{d\sigma_{el}}{dt}\right]_{t=0} = \frac{\pi}{k^2}\left[\frac{d\sigma_{el}}{d\Omega_{CM}}\right]_{\theta_{CM}=0} = \frac{\pi}{k^2}\left|\mathrm{Re}\, f_{CM}(0) + \mathrm{i}\,\mathrm{Im}\, f_{CM}(0)\right|^2. \quad (50)$$

Introducing $\rho = \mathrm{Re}\, f_{CM}(0)/\,\mathrm{Im}\, f_{CM}(0)$, we rewrite (50) as

$$\left[\frac{d\sigma_{el}}{dt}\right]_{t=0} = \pi\left|\frac{(\rho+\mathrm{i})\,\mathrm{Im}\, f_{CM}(0)}{k}\right|^2 = \pi\left|\frac{(\rho+\mathrm{i})\sigma_{tot}}{4\pi}\right|^2 = \frac{(1+\rho^2)\sigma_{tot}^2}{16\pi}, \quad (51)$$

where we have used the optical theorem, $\sigma_{tot} = \frac{4\pi}{k}\mathrm{Im}\, f_{CM}(0)$, where k = the center of mass momentum, in the next-to-last step. Integrating (49) over t from $-\infty$ to 0, we find the total elastic scattering

$$\sigma_{el} = \frac{\sigma_{tot}^2(1+\rho^2)}{16\pi B}. \quad (52)$$

We can finally rewrite (52) in the useful form

$$\frac{\sigma_{el}}{\sigma_{tot}} = \frac{\sigma_{tot}(1+\rho^2)}{16\pi B}. \quad (53)$$

It should be pointed out that the application of (53) to the γp and $\gamma\gamma$ processes assumes that the photon has turned into a hadron and is interacting hadronically. Here, the term “elastic” implies reactions such as $\gamma + p \to V_i + p$ or $\gamma + \gamma \to V_i + V_j$, where $V_i = \rho, \omega, \phi$, i.e., the photon has turned into a vector meson that elastically scatters.

Applying to (53) the fundamental condition used by Block and Kaidalov [1] – the ratio of elastic to total cross section is process independent – after using the equality of the ρ values for all three processes, we find that

$$\frac{\sigma_{nn}}{B_{nn}} = \frac{\sigma_{\gamma p}}{B_{\gamma p}} = \frac{\sigma_{\gamma\gamma}}{B_{\gamma\gamma}}. \quad (54)$$

Utilizing the cross section factorization theorem of (1), $\sigma_{nn}(s)/\sigma_{\gamma p}(s) = \sigma_{\gamma p}(s)/\sigma_{\gamma\gamma}(s)$, in (54), we deduce the factorization theorem of (2) for the nuclear slopes B, i.e.,

$$\frac{B_{nn}(s)}{B_{\gamma p}(s)} = \frac{B_{\gamma p}(s)}{B_{\gamma\gamma}(s)}. \quad (55)$$

In conclusion, we have demonstrated the robustness of the factorization theorem for total cross sections for nn, γp and $\gamma\gamma$ scattering, $\frac{\sigma_{nn}(s)}{\sigma_{\gamma p}(s)} = \frac{\sigma_{\gamma p}(s)}{\sigma_{\gamma\gamma}(s)}$, even when using a monopole form factor for the photon and a dipole form factor for the nucleon. Further, we have proved the factorization theorem for the nuclear slopes, $\frac{B_{nn}(s)}{B_{\gamma p}(s)} = \frac{B_{\gamma p}(s)}{B_{\gamma\gamma}(s)}$, along with the theorem $\rho_{nn} = \rho_{\gamma p} = \rho_{\gamma\gamma}$ of Block and Kaidalov [1] using the more general conditions of analyticity along with the optical theorem. Experimental evidence for these theorems can be found in [6–8].

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